The distortion of knots

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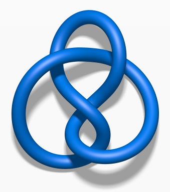
Knots

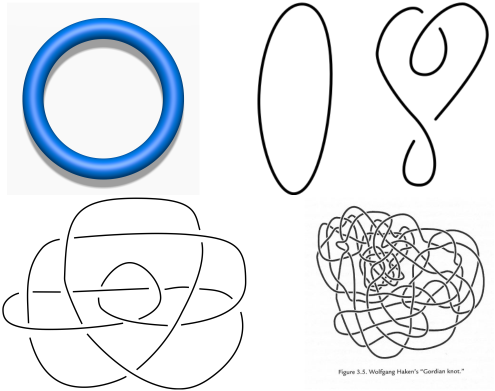
 

Two views of the figure eight knot.

Origins of knot theory can be traced back to Lord Kelvin who thought different elements corresponded to different knots (1860s).

First knot tables (1885) compiled by physicist Peter Guthrie Tait.

Knots

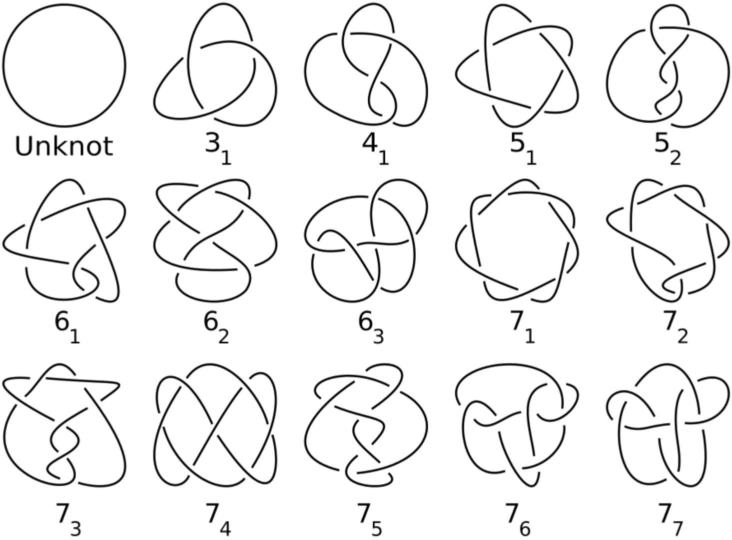


Five views of the unknot.

Definition

A *knot* is a map $\gamma : S^1 \to \mathbb{R}^3$ up to homotopy through smooth embeddings.

Knot table



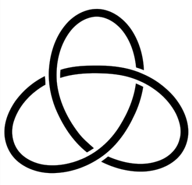
Distortion

Definition (Gromov)

The *distortion* of a (unit speed) curve $\gamma : S^1 \to \mathbb{R}^3$ is:

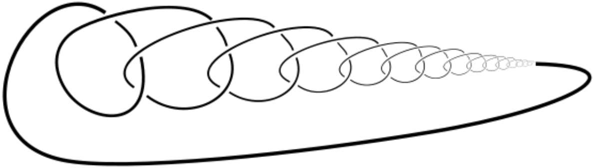
$$\delta(\gamma) := \sup_{s,t \in S^1} \frac{|\gamma(s) - \gamma(t)|}{|s - t|} \geq 1 \tag{1}$$

The distortion of a knot K is the minimal distortion among all curves γ in the knot class K.



Numerical simulations indicate that $\delta(\text{trefoil}) < 7.16$.

Knots with small distortion



There are rather wild knots with finite distortion.

Curves with finite distortion can have infinite total curvature.

Thus distortion is a very weak measure of complexity.

Distortion of knots

Theorem (Gromov)

For any closed curve γ, we have $\delta(\gamma) \geq \frac{1}{2}\pi = 1.57\ldots$. Equality holds if and only if γ is a round circle.

Corollary (Gromov)

$\delta(\mathit{unknot}) = \frac{1}{2}\pi = 1.57\ldots$.

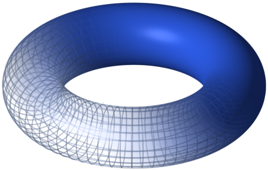
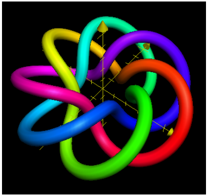
Theorem (Denne–Sullivan)

For any knotted closed curve γ, we have $\delta(\gamma) \geq \frac{5}{3}\pi = 5.23\ldots$.

Corollary (Denne–Sullivan)

$\delta(K) \geq \frac{5}{3}\pi = 5.23\ldots$ *for* $K \neq$ *unknot.*

Torus knots

A torus and the torus knot $T_{3,7}$

Lemma

The standard embedding of the torus knot $T_{p,q}$ in $\mathbb{R}^3$ has distortion $\gg \max(p, q)$.

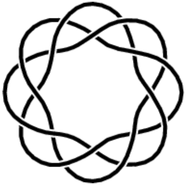
Question (Gromov)

Are there knots with arbitrarily large distortion? Specifically, is it true that $\delta(T_{p,q}) \to \infty$?

Distortion of torus knots

Theorem (P)

$\delta(T_{p,q}) \geq \frac{1}{160}\min(p,q)$ *for torus knots* $T_{p,q}$.



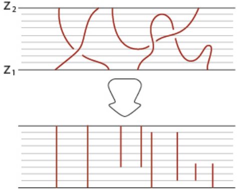
Torus knot $T_{3,8}$.

Proof that torus knots have large distortion

Ingredient 1 (integral geometry):

$$\int_{-\infty}^{\infty} \#(\gamma \cap H_t)\, dt \leq \text{length}(\gamma) \tag{2}$$

where H_t is the hyperplane $\{(x, y, z) \in \mathbb{R}^3 : z = t\}$.

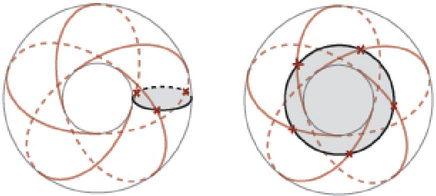


Proof that torus knots have large distortion

Ingredient 2:
Let $\gamma \subseteq T \subseteq \mathbb{R}^3$ be the (p, q)-torus knot.
Given a family of balls $\{B_t\}_{t\in[0,1]}$ with $\#(\gamma \cap \partial B_t) < \min(p, q)$ for all $t \in [0, 1]$, we have:

$$g(B_0 \cap T) = g(B_1 \cap T) \tag{3}$$

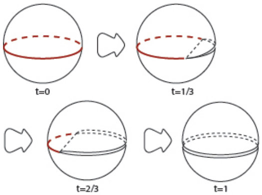
Key point: $\#(\gamma \cap \partial B_t) < \min(p, q)$ implies that $\partial B_t \cap T$ is inessential in T.



Proof that torus knots have large distortion

Suppose $\gamma \subseteq T \subseteq \mathbb{R}^3$ with $\delta(\gamma) \ll \min(p,q)$.

Take any ball $B(r)$ of radius r such that $g(B \cap T) = 1$.
Ingredient 1 $\implies$ there exists $r \leq r' \leq \frac{11}{10}r$ such that $\#(\gamma \cap \partial B(r')) \ll \delta(\gamma) \ll \min(p,q)$. Similarly, find a disk cutting B in half (approximately), and intersecting γ in $\ll \min(p,q)$ places.



Ingredient 2 $\implies$ T intersected with upper or lower half-ball has genus 1.

We have thus produced a smaller ball with the same property $g(B' \cap T) = 1$! (contradiction)

Questions

Question

Is it true that $\delta(T_{2,p}) \to \infty$ as $p \to \infty$?

L. Studer has shown that $\delta(T_{2,p}) \ll p/\log p$ (the standard embedding of $T_{2,p}$ has distortion $\asymp p$).

Question

Is it true that $\delta(T_{p,q}\#K) \to \infty$ as $p, q \to \infty$ (uniformly in K)?

Outlook

Topology	$\longleftrightarrow$	Geometry
manifolds, knots, etc.	$\longleftrightarrow$	curvature, distortion, etc.

Credits

Pictures taken from:

- Wikipedia
- http://conan777.wordpress.com/2010/12/13/knot-distortion/

Thank you