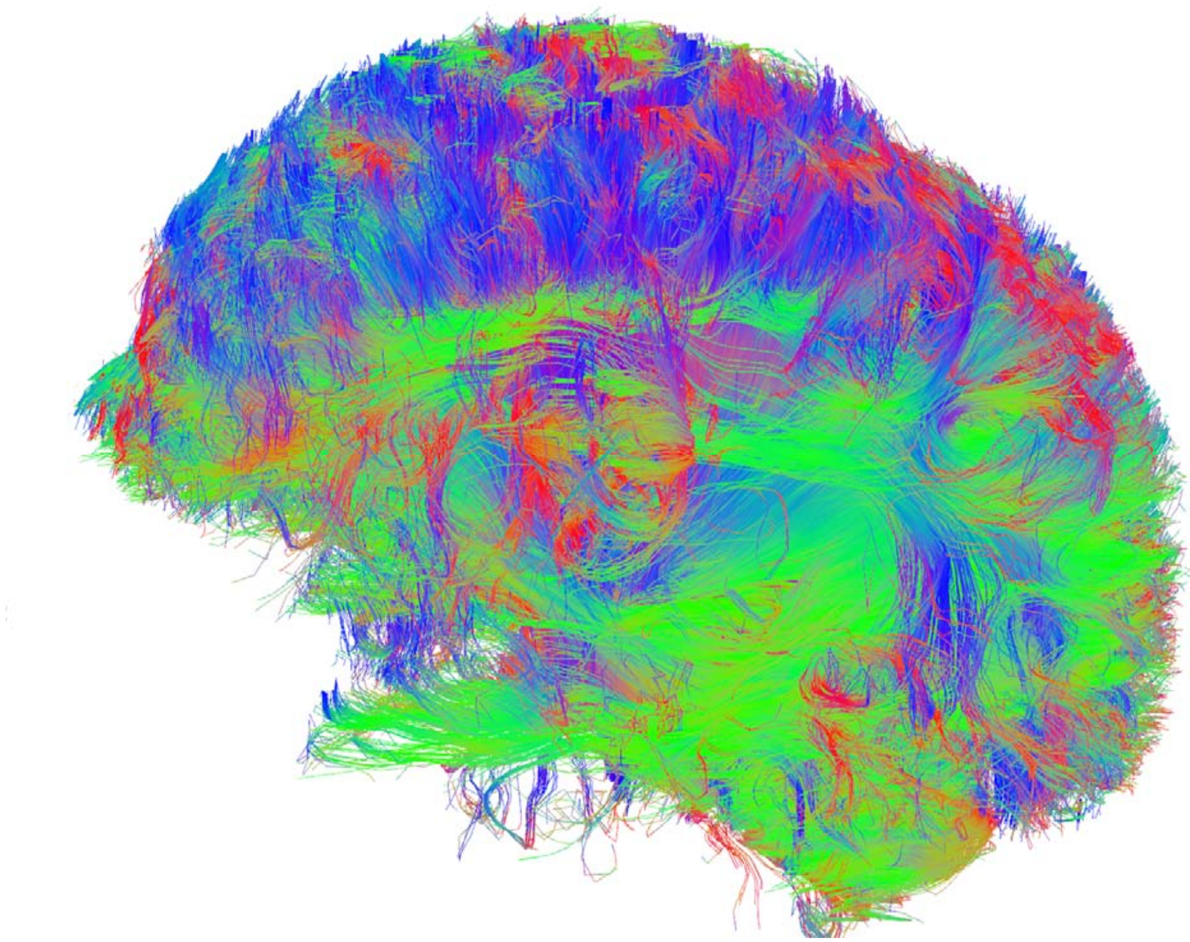

Workshop Report:
NSF Workshop on Neuroscience and Mathematical Cognition

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NSF Neuroscience and Mathematical Cognition Workshop

NSF Workshop on Neuroscience and Mathematical Cognition

Executive summary

Advances in the technology for detecting, measuring, and imaging brain activity during cognitive-task performance offer new opportunities for expanding our understanding of mathematical cognition and for providing guidance to mathematics education. Brain imaging is a novel and independent source of evidence that can help in answering challenging questions about mathematical cognition, with potential implications for education.

Recent research suggests that three areas of the parietal cortex of the human brain are especially important in the performance of very elementary numerical tasks and seem to have distinct functions: First, a general appreciation of the magnitude of quantities involves activity of the horizontal segment of the intraparietal sulcus (HIPS). Second, verbal processes in math, such as the use of multiplication facts, involves activity in the left angular gyrus, near well known language areas of the brain. Finally, spatial representation of a number line and/or spatial control of attention to a number line involves activity of the superior posterior parietal cortex on both sides of the brain.

To date, research exploring brain activity related to mathematical thinking has focused on elementary number concepts and arithmetic operations and some work has addressed simple math problem solving and algebra. Mathematics is a large and complex field, and a research agenda representing the broader field of mathematical cognition can now be anticipated. Several promising areas of investigation have been identified:

- **Early Development of Mathematical Thinking:** Although cognitive neuroscience research to date has concentrated heavily on this topic, important open questions remain. For example, we do not know how the verbal symbolic number systems becomes more effectively connected to the HIPS representation of non-verbal magnitude during development and what aspects of the HIPS-related circuitry occur in non-human animals.
- **Statistics and Probability:** Both humans and non-human animals show considerable informal appreciation of probabilities in their everyday lives, but formal education in probability and statistics is not common in the early years, possibly representing a lost opportunity. Changes in brain activity and learning, as a result of formal education, is a promising area of study.
- **Spatial Aspects of Mathematics:** Although the importance of spatial thinking to mathematics is generally accepted and especially evident for geometry and analytic geometry, questions regarding the neural bases of these abilities have not yet been addressed. Given that a great deal is already known about the brain areas involved in spatial processes, thus this is a particularly promising topic for investigation.
- **Executive and Planning Functions:** Executive and planning functions are very important to doing mathematics, especially in solving complex problems and formulating proofs. Our understanding of the brain areas and brain activity involved in such functions is improving rapidly, making this an important and promising topic for investigation. Explanations for the relationship between measures of general intelligence and math achievement may lie here, as well as an understanding of the proper role of practice in math learning.
- **Affective Dimension of Mathematics Learning**: Mathematics phobia is a well-known phenomenon, but many people also like mathematics and experience a kind of cognitive joy in solving mathematical problems. Careful investigation of the impact of math phobia on mathematical cognition might also provide more general insight into interaction between affect and cognition, such as cognitive performance under the stress of war or other emergencies. A great deal is known about the brain areas involved in emotions, providing a substantial foundation for such studies.

The full potential of this research will involve methodological advances on many fronts, relevant training opportunities, and multidisciplinary research teams. It is somewhat surprising that these rich areas have been so little explored. The time is ripe for research aimed at realizing this potential.

NSF Neuroscience and Mathematical Cognition Workshop

I. Introduction

Recent advances in the technologies for detecting, measuring, and imaging brain activity during cognitive task performance offer new opportunities for advancing our understanding of mathematical cognition and for providing guidance to mathematics education. Brain imaging data is a largely novel and independent source of evidence that can help in answering challenging questions about mathematical cognition. The time is ripe for a research program realizing this potential. In May of 2007, NSF convened a workshop to further explore these opportunities and identify the most important research questions and most promising research directions. This is the report of that workshop.

Recent research, discussed by John Anderson, has identified three distinct brain areas that are involved in performing elementary mathematical tasks, all within the parietal cortex of the human brain. One known as HIPS (shown in pink in the figure) – represented on both sides of the brain – is active in tasks calling for an appreciation of magnitude, such as judging whether one symbolic number is larger than another, or one collection of objects larger than another. Behavioral evidence suggests this same system is present in the brains of many animal species, and recent results recording the activity of single neurons in monkey brains suggests that homologous areas have similar functions in monkeys. A second area, located on the left angular gyrus (green in the diagram) near well known language areas of the brain, is involved in doing mathematics that uses memorized verbal facts. A third bilateral area of the superior parietal lobe (shown in blue) is involved in spatial processing in mathematics, such as the use of a number line to do subtraction.

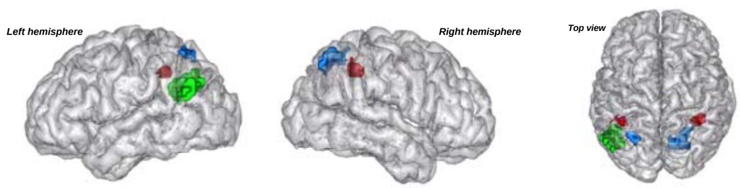

Figure 1

Already there are some hints of the value that such new understanding of early development may have for mathematics education. It seems that there may be a relationship between small size of the HIPS area and the occurrence of dyscalculia, an extreme difficulty in learning elementary mathematics. Intensive training on numerical comparisons, presumably exercising the HIPS, has been reported to improve learning in dyscalculic children. This effect would be parallel to other examples of neural plasticity in response to intensive practice, as has been found in motor skills. Workshop participant Bruce McCandliss suggested that a research program parallel to recent work on dyslexia could be pursued in the domain of early mathematics: using measures of neural activity and connectedness to identify possible deficits leading to learning difficulties in mathematics and then testing very targeted instructional interventions suggested by the neural findings. In the case of dyslexia, such instructional interventions are already being compared to more usual instructional practices in controlled educational experiments.

Results obtained by Ansari and his collaborators suggest that both symbolic and non-symbolic number processes involve the HIPS area but that symbolic number processing in this area is a late development; in children, the frontal lobe is more involved. It remains to determined whether the HIPS area itself is underdeveloped in individuals with dyscalculia, or whether there is inadequate development of neural connections to this region.

Most research to date has focused on very elemental number concepts. However, John Anderson has shown how a powerful triangulation of behavioral training research methods, computational cognitive modeling and imaging of brain activity can illuminate considerably more advanced mathematical learning at the level of high school algebra. He has developed the capability to model human cognitive performance and learning at a very fine-grained level of detail. Combining these models with what is known about which brain areas seem to support various cognitive functions, he has shown that it is possible to predict the brain activity which will be observed from models based on behavioral data. Often the predictions are strikingly accurate, but when there is a mismatch, the resolution can advance both the analysis of cognitive functions at the level of cognitive theory and our understanding of the functions performed by areas of the brain. Current descriptions of those brain functions are typically quite crude because they are based on behavioral tasks that do not reflect modern cognitive psychology and otherwise limited by the problems of working with patients suffering brain damage. In the accompanying figure

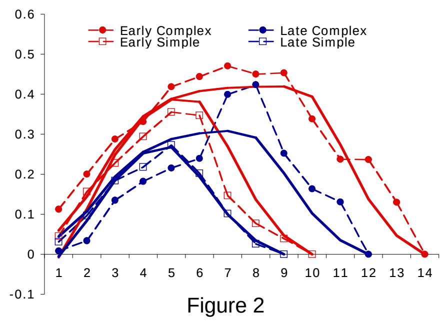

Figure 2

John Anderson (unpublished data), shows the predicted and empirically observed effects of practice solving linear equation problems on brain activity in the prefrontal cortex, which is involved in retrieval of information from memory. (The figure shows the mean percent change in BOLD signal in the left lateral inferior prefrontal cortex. Separate curves are shown for students performing simple and complex algebraic transformations early and late in the training. The dotted lines connect the observed data and the solid lines give the predictions of a model that assumes this activation reflects retrieval from declarative memory.) Simple and complex transformations of equations are contrasted in this graph. Most cognitive tasks, even those that seem very simple, involve many different cognitive functions that use many different areas of the brain. Refined cognitive modeling, good control of task performance and its timing, and imaging of brain activity can work in tandem to provide rapid advance in both our understanding of cognitive functions and our understanding of brain function.

II. Finding a Psychologically Valid Analysis of Mathematical Skills

Even linear equation problems, though much more advanced than adding, subtracting, multiplying or comparing two single digit numbers, are not very advanced mathematics. As both a preliminary foundation and a research agenda, we must consider the structure of the whole domain of mathematics. This problem requires multiple perspectives. Professional mathematicians are of course the best authority on the analysis of mathematics, including the important distinctions among different varieties of mathematical thinking and the relations among different areas of mathematics. We now know, however, that techniques developed in cognitive psychology/science assist in revealing the structure of knowledge and skill in their domain. Mathematicians' conscious analysis of their field may not be the same as an analysis based on cognitive psychology that will prove effective for teaching and learning mathematics. On the other hand, mathematicians have penetrated to the fine structure of intuitive mathematical concepts such as natural numbers (1, 2, 3 …) and counting in a way that has proved fruitful in guiding research on the development of early mathematical understanding, as demonstrated in the research of Rochel Gelman. Traditional neuropsychology, based on examining the cognitive effects of brain damage, has provided many surprising insights into the structure of perceptual and cognitive functioning – for example, there is a surprising dissociation between subtraction and multiplication of simple numerals: one of these functions may be retained after injury while the other is lost. A great attraction of modern techniques for imaging brain activity is that they can enable us to obtain such evidence in a systematic way, freeing us from dependence on the vagaries and tragedy of brain damage. For example, it now seems that subtraction evokes activity in the third of the parietal areas (PSPL) shown in Figure 1, perhaps because of processes involving the use of a spatial number line, whereas multiplication involves the use of the second, verbal area, probably because of reliance on memorized multiplication tables. Seeing activity in different brain areas as these different operations are performed aids our analysis of these elementary mathematical skills. Such evidence, as well as instructional experiments, may show us that more complex mathematical skills are not built in the way that we had thought, yielding new insights to guide instructional approaches in mathematics.

In addition to the types of evidence already mentioned, advanced psychometric techniques, such as those Curtis Tatsuoka presented at the workshop, can be used to analyze the structure of mathematical knowledge and skill. Here, one can examine the ordering or partial ordering of *attributes* of mathematical tasks. Typically, attributes are units of analysis much smaller than a single mathematics problem but much larger than the units involved in Anderson's style of modeling.

In a new research program, it would be helpful to sketch out the larger structure of mathematics, to ensure that we are not ignoring major aspects – for example, it seems that little attention has been given to the neural basis of geometrical thinking, despite the obvious involvement of spatial thinking in this area of mathematics and our considerable understanding of the areas of the brain involved in understanding, thinking about, and moving in space. Rochel Gelman mentioned rational numbers, which are not very advanced mathematics but represent a major point of mathematics learning difficulty for many people. Additional cognitive neuroscientific research on algebra, algebra word problems, and arithmetic word problems as well, would be highly valuable. At the workshop, Paul Cobb suggested that statistics and probability would be promising subject matter for looking at changes in the brain related to learning because these were late developments in the history of mathematics and, and most adults have not had formal school training in these areas. William McCallum suggested several topics that might be investigated with neural techniques. One topic is distinguishing between seeing mathematical expressions as mere visual patterns and seeing them as structured expressions. It is likely that different areas of the brain are engaged when this transition is made, very likely areas associated with verbal processing, including the processing of syntax. He also raised a question about the processing underlying the ability to make quick estimates of the answers to problems or of some features of the answers, like whether they will be positive or negative. Most intriguing but most speculative was his suggestion that mathematicians come to see mathematical objects as somehow *real*, and that one might be able to find some neural correlate of this reification process. This is a reminder of a point also made by workshop participant Alan Yuille: the mathematics addressed by this research to date barely touches on the complexity of mathematics as a whole.

III. Early Development of Mathematical Thinking

III.a. HIPS: The "core quantity system"?

A great deal of recent research has shown that humans (including human infants) and many non-human animals share a capacity for appreciating numbers and numerosity. This system does not provide an exact linear representation of numbers but a more logarithmic one, obeying Weber's law – the noticeable difference in numerosity grows with the number. Even when people are explicitly asked to compare numbers presented symbolically, the time required to do the comparison shows an effect of distance between the numbers that behaves in this proportional way. For example, adults are much faster at responding that 9 is greater than 7 than they are at responding that 99 is greater than 97. In humans, exercise of this capacity involves activity in the HIPS (horizontal segment of the intraparietal sulcus) area of the brain. A corresponding area with comparable function has been found in the rhesus macaque money. Neurons responsive to specific numbers have been found in both the inferotemporatl cortex of the monkey – an area that supports working memory, and in the area analogous to the human HIPS. The fact that this capacity is shared with non-human animals opens the possibility for neuroscientific investigation of the neural network s operating in this area, coupled with computational modeling of those networks, in order to understand how those representations work to perform the tasks. Another phenomenon that might be investigated is *subitizing*, the capacity to appreciate the exact value of small numbers: one to three or four.

Behavioral research with human infants and children suggests that the HIPs mediated system for appreciating numerical values as magnitudes that are proportional to number emerges early in development. Human infants can detect changes in numerosity, and their ability to detect numerical changes is dependent on the ratio of the two values being compared. This indicates that Weber's Law holds even in infancy. Recent fMRI studies show that by as early as 4 years of age children and adults show overlapping IPS activity when they passively view changes in numerosity. A critical question is how this nonverbal Hips mediated system that appears to be shared by adults, human infants, and nonhuman animals comes to interface with verbally mediated systems for symbolic number representation over development. As noted above, making the connection between abstract symbols for numbers, such as Arabic numerals, and the HIPS area, seems to require a substantial period of learning and/or development. Tensor diffusion imagery (DTI) might be used to assess the connectivity between the frontal lobe areas that seem to be processing number symbols in young children and the HIPS area. Both fMRI and DTI might be used to track changes in the brain that occur in response to deliberate training efforts to develop more efficient associations between number symbols and the more primitive or elemental system for encoding quantity or magnitude.

Perhaps even more interesting would be exploration of the much broader role that the HIPS system may play in mathematical cognition and our understanding of a dimensionalized world. There is a large body of older psychological research called psychophysics that has explored the perception of magnitude in many perceptual domains, notably the work of S.S. Stevens and his followers. People can provide numerical estimates of the length of lines, the brightness of light, or the loudness of sound in a way that provides surprisingly good, if often non-linear data. In addition, they can perform cross-modality matching tasks, matching the brightness of a light to the loudness of sound. Quite possibly the same HIPS system supports all of these forms of magnitude. It would certainly be interesting to know. Furthermore, this could illuminate how mathematics has meaning as we apply it to the reality represented by these perceptual dimensions. Given the apparent qualitative or approximate nature of the HIPS representation, it may also prove central to numerical estimation skills. Since the advent of calculators, estimation skills have become a problematic issue in mathematics education. To use calculators effectively, one must have a sense of the reasonable approximate result, in order to detect errors.

III.b. The verbal - symbolic aspect of mathematics: from rote learning to abstract pure mathematics

The second area of parietal cortex that seems to be involved in mathematical thinking is thought to be involved in verbal mathematical processes. It is near areas known to be language processing areas. As mentioned above, damage to this area has been found to interfere with simple multiplication. Most people rely upon memorized multiplication tables in order to perform such computations, an essentially verbal process. Addition also tends to involve memorized relationships but there is evidence of mixed strategy use. Developmentally, of course, one sees a shift from early strategies that involve counting the members of both sets to be added or counting on from the value of the first numeral for the number of units represented by the second numeral. As addition facts are learned or memorized, a shift to reliance on memory is observed in behavioral data. (Cf. the work of Robert Siegler). Presumably we could observe evidence of this strategy shift in brain activity as well. Multiplication can also be handled by a strategy analogous to counting on, repeated addition. This strategy does require keeping track of the number of additions being made and may therefore tax processing capacity, but there may be individuals who use it.

Neuropsychological and imaging studies of simple arithmetic operations only begin to tap the likely use of essentially verbal strategies in mathematics. In his study of algebra, Anderson reported activity in the left fusiform area, close to the well known word form area, that may be involved in parsing mathematical expressions. In general, symbol manipulations governed by mathematical rules have considerable similarity to grammatical language processing and may invoke the same or nearby areas in the brain: this would be a fruitful area for investigation. Much of pure mathematics of the most advanced character is largely symbol manipulation within rule systems, seeking proofs. It would be interesting to see how and where in the brain this processing seems to be done.

III. c. A mental number line, an elementary form of spatial thinking in mathematics

The relationship of mathematical thinking to human spatial abilities is a cliché. Yet, current imaging studies have barely touched upon the spatial aspect of mathematical thinking, despite the fact that a great deal is known about the areas of the brain subserving spatial perception, cognition and motor action. The third area (PSPL) identified in Figure 1, has been claimed to have a spatial processing function in mathematics on the basis of items of weak evidence: The PLSP area is near other brain regions known to mediate spatial processing; and speculation that simple subtraction (which activates PLSP) may rely on the use of the mental number line. A further step beyond current work would be to seek evidence that this area is in fact the locus of a number line representation. One approach would be to employ a proven educational program for very early mathematics that uses game-like activities to develop a strong number line representation, including successor and predecessor relationships among numbers, addition and subtraction as movements along the number line (Case et al., 1999). Would we see brain activity and changes in brain activity associated with this training in this putatively spatial area or in the HIPS area, or both? How would these results compare with the effects of current training studies that the Dehaene group is doing to promote development of the HIPS area?

IV. Spatial Aspects of Mathematics

Studies of geometry and geometrical thinking are the obvious direction to go if one wants to begin understanding the role of spatial thinking in mathematics. Although it is unclear whether measures of human spatial ability predict mathematical achievement in general any better than do measures of general academic ability or even measures of verbal ability, there is at least one study showing that measures of spatial ability do strongly predict achievement in geometry and analytic geometry when it is measured by course specific tests. Mental rotations and translations are likely involved in discovering and planning geometrical proofs, for example. These mental operations have been much studied in basic psychological research, and they have also been shown to be susceptible to training effects. Thus, there is a substantial foundation for taking these investigations into neuroimaging research.

Graphs of data have a strong claim to be the first language of science, and the reading of graphs and charts is part of the standard mathematics curriculum. Again, mental translations and other mental transformations of a spatial character may be involved in reading graphs or using them to discover new relationships in data. Graphs can be said to be built of multiple number lines (2, 3 or perhaps even more, depending on the dimensionality of the graphic.) So, it would be interesting to see how the cognitive processing of scientific graphs relates to activity in that third parietal area thought to be representing a number line. Today, computerized visualization of data is increasing popular as an approach to scientific discovery. These graphics permit dynamic exploration through operations such as rotation and one can “rotate” into dimensions beyond the third in order to explore high dimensional data. It would be interesting to explore how the resources of the human brain are drawn upon to effectively use these sophisticated graphical representations.

It is important to remember also that interacting with even the most conventional static representations of mathematics calls upon “spatial abilities” – the deployment of spatial attention in order to read mathematical expressions, diagrams and graphs. Additional spatial processing may be seen when people comprehend the content of mathematics word problems and built a mental model of the situation described. Often these have a spatial character. An obvious and simple example would be route problems of the type involving two vehicles traveling at different speeds and meeting or overtaking each other. Most people will build a mental image of this situation.

Obviously there is a great deal to be learned about the spatial processing in the brain as people do many different kinds of mathematical thinking, and there is a very substantial foundation of prior related research to build upon in these investigations.

V. Probability and Statistics

Probability and statistics are areas of mathematics of high practical importance for making rational personal and societal decisions. For several reasons, this is an area ripe for investigation. There is a large body of research showing that both humans and non-human animals are very sensitive to probabilities of reinforcement of their behavioral choices. Thus, this is a potential research area that is somewhat parallel in opportunities to the work on early number development. However, there is the interesting difference that most adults have not had formal school training in these areas and have difficulty dealing with problems in probability and statistics even when they are engaged in formal education on these topics. There is the potential research advantage that most adults have not mastered the formalizations of probability and statistics and would still be operating, most likely, with whatever primitive system is common to both humans and non-human animals. We might be able to observe changes in brain activity that occur as a result of formal instruction, uncomplicated by simple maturation going on at the same time.

VI. Executive and Planning Functions in Mathematics

Doing and learning mathematics certainly call upon the executive functions of the brain. This is true in relatively trivial ways – response selection and execution – even when the task is an overlearned simple addition problem. Trivial as this may seem, it will nevertheless show up in measured brain activity. More complex mathematical problem solving and proof making further tax the executive functions of the brain. Anderson, for example, presented data on activity in the anterior cingulate cortex during performance of an algebra task. FMRI research has revealed that the anterior cingulate cortex is very much part of the executive processing system of the human brain, a novel insight provided by this method. Recent research for example has also shown that activity in the executive control areas of the human brain becomes much attenuated as a skill becomes *automated* with practice (Hill and Schneider, 2006). Automating skills is thought to free up executive processing capacity for other concurrent task demands or to cope with more complex problem solving demands that call upon the automated skills. Obviously this notion is related to controversies in mathematics education about the need for or appropriateness of drill and practice activity.

As we aim to understand how the transition from controlled to automated skill occurs, there is much to be discovered about the nature of natural learning processes. Drawing upon experience in the field of machine learning, as well as research on effective instructional practices, workshop participant Ken Koedinger suggested that different brain areas may train each other, yielding self-supervised learning. Several theorists have proposed theories of learning that are hybrids of connectionist/neural net learning and symbolic learning. In these models, initial rule-based performance can "train" neural nets or neural net learning can be consolidated and summed up into symbolic rules. It is well known that instructional techniques that use multiple representations, thus engaging multiple neural systems, often facilitate learning. Executive control or learning strategies may be involved in coordinating these multiple representations, using one of them as a check on the other, for example. While Koedinger emphasized the idea of several systems working in concert to produce effective learning, Russell Poldrack suggested, based on research he has done on two systems involved in memory, that sometimes competing systems may yield the most effective learning.

It is well known in mathematics education that multi-step word problems are very difficult. In fact, they are rarely presented in the school curriculum. The solution process must be planned, and the problem solver must also keep track of the solution process. These are both executive control functions. In more advanced mathematics, proof making imposes similar but greater demands. At the workshop, Sharlene Newman presented brain imaging research on planning and execution monitoring functions in the Tower of London task. This task is similar in character to mathematical tasks, with the number of necessary steps in the solution being a prime indicator of difficulty. Her results to date suggest that the right prefrontal cortex is involved in planning the solution, whereas the corresponding area on the left may be more involved in monitoring execution of the solution. Furthermore, the amount of activity on the right increases with problem difficulty and is related to the person's measured working memory capacity – individuals with lower working memory capacity show more right frontal activity. (It is noteworthy that activity in a brain area does not necessarily reflect more effective processing: In many instances it has been demonstrated that the brain volume used to perform a task decreases with learning and/or ability.) Measures of working memory capacity, particularly working memory for goals, are closely related to established traditional measures of general intelligence, such as the Raven Progressive Matrices test (Carpenter et al., 1990), and those measures are very strong predictors of mathematical achievement.

Perhaps a great deal could be gained in mathematics education by research similar to Newman's that probes the mechanisms underlying that correlation between general intelligence and mathematics learning. Perhaps instruction could be designed to reduce the demands on working memory of the instruction itself. Perhaps the judicious use of practice could be used to build "chunks" of mathematical problem solving skill that reduce those demands, enabling more students to succeed in more advanced mathematics.

Executive and control functions are quite subtle and difficult to isolate in any test or assessment of cognitive function. To do so, it is likely to be necessary to have a good, detailed computational model of task performance, as is true in both Anderson and Newman's research. Neuropsychological case reports of the effects of injury to parts of the frontal lobes are often quite fuzzy but clearly describing defects of executive function. The human frontal lobes are proportionally much larger than the corresponding areas the brains of related animals studied in neuroscientific research, such as the rhesus monkey. Consequently, research on these functions seems one of the most promising areas for fruitful interaction between cognitive scientists doing detailed computational modeling of cognition and brain imaging researchers. Furthermore, mathematics provides very well defined cognitive tasks that tax executive processing, and are thus usable in such research.

Another aspect of frontal lobe functioning should be of interest in relation to mathematics education. Full development of frontal lobe function appears to continue throughout adolescence; myelination of the nerve tracts continues throughout that period. This may have important implications for mathematical cognition. At the workshop, Elizabeth Brannon suggested using Diffusion Tensor Imaging (DTI), which reveals the neural connections among brain areas, to examine this development. (A DTI image is shown on the cover of this report). She showed an example of the relationship between maturity of the brain connections as shown by DTI (in this case corpus collosum connections between the two parietal lobes) and a type of mathematical cognition. For seven-year-olds, the degree of connectivity measured by variable is strongly related to performance on numerical comparison tasks, but not for adults. As seen in the figure, performance on number tasks correlated with FA (fractional anisotropy) values along Isthmus (parietal), but not other tracts in children. No correlations were found in adults. Cantlon, Brannon, and Pelphrey, unpublished work in the laboratory; manuscript in preparation.The accompanying brain image of the brain shows callosal fiber tracts in children. In orange are callosal fiber tracts connecting parietal cortices (Isthmus). In green are callosal fiber tracts connecting frontal cortices (Genu). In blue are callosal fiber tracts connecting occipital cortices (posterior splenium). Workshop participants Vinod Menon and Bruce McCandliss also endorsed the importance of looking at brain maturation in relation to mathematical learning, and the use of DTI for that purpose.

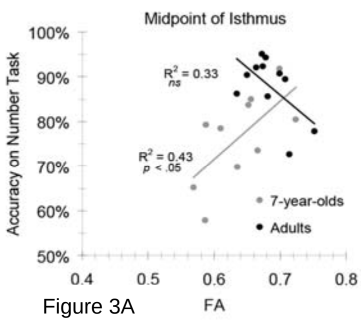

Figure 3A

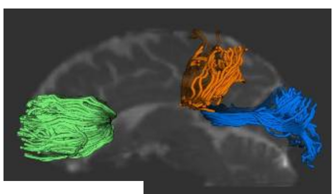
Figure 3B

Perhaps we will discover that appropriately designed mathematics instruction can accelerate the development of relevant nervous connections, or perhaps we will discover that it is best to wait until the brain is ready to meet the demands of more advanced mathematics. Surely there must be individual differences in the timing of this development that will have implications for success in mathematics learning.

VII. The Affective Dimension of Mathematics: Phobia and Cognitive Joy

It is well known that *math phobia*, or mathematics related anxiety, is a common problem. At the workshop, Peter Bergethon suggested this as an appropriate topic for research on brain activity related to mathematics learning. A great deal is known about the emotional circuitry of the brain, and there is a strong background of behavioral research on mathematics anxiety, so this is indeed a promising area for immediate investigations. Well-developed efficient questionnaires are available to measure mathematics anxiety and experimental manipulations like the invocation of *sterotype threats* have been shown to have significant effects on mathematical performance. In addition to the educational significance of mathematics anxiety and phobia, there is an important more general scientific contribution to be made by pursuing this line of research. We are certain that anxiety and fear have important impacts on cognitive function, but little is known about the exact nature of those effects. Investigations of mathematics anxiety could provide a good start on that more general scientific problem. They will help us understand human performance in stressful, anxiety provoking situations in both the military and civilian life.

Although quite a few people fear mathematics, that is not the whole story. Many people like mathematics and will name it as their favorite school subject. Surprisingly to many people, females are just as likely as males to like mathematics. For many, there is a form of cognitive joy to be found in solving mathematics problems. Early German psychologists observed a general phenomenon in human development they called *funktionslust*, an intrinsic motivation to exercise newly developed capacities. In mathematics, this may be seen in adolescent mathematical prodigies, where the newly developed capacity would be formal operational thinking. Pure mathematics is surely the prime example of formal operational thinking. For mathematics education in particular, for the development of the field of mathematics itself, we may want to be sure that the opportunity presented by *funktionslust* is not lost: the right educational opportunities must be available at the right time.

Perhaps a clever experimental design could enable us to see activity in the known pleasure centers of the brain that is associated with positive emotion in mathematical problem solving and proof development. Seeing the obvious reality of this phenomenon could have important consequences for mathematics education that it is difficult to anticipate. Like the recent finding of pleasure center activity associated with altruistic acts, it could well alter our thinking about mathematics education.

VIII. Enabling Methodological Developments

Realizing the full potential of this exciting research area will be enabled by continuing methodological advances and by mechanisms for sharing those advances as they are achieved. Each technology for detecting and measuring brain activity imposes limitations on the cognitive and behavioral activity that can be studied. Currently available methods for measuring signals associated with cognitive activity are typically weak and noisy, requiring averaging over many trails and/or research participants. If one wants to study cognitive flexibility or fluid intelligence in approaching novel problems, one cannot have subjects highly practiced on the problems. By using creative experimental paradigms, psychologists have been able to learn a remarkable amount about the cognitive capabilities and some of these tools can be applied to the study of mathematical thinking and learning. Workshop participant Layne Kalbfleisch provided an interesting example in an fMRI study: Encouragingly, she succeeded in identifying both cerebral and cerebellar brain areas supporting cognitive flexibility in solving non-verbal problems similar to those in the Ravens' Progressive Matrices test. This may be the methodological exception that proves the rule.

Imaging and recording methods: Curently available noninvasive imaging technologies sum up activity over several seconds, a long time by the standards of cognitive activity, and with a significant time lag – because what is actually being detected is blood flow correlated with neural activity. Therefore, matching the brain activity that is observed this way to what is actually going on in the participant's cognition is challenging. In addition, there is a need for benchmarking (as well as adjustments for the differences in the shapes of individual human brains) in order to support the averaging of data across several participants. In order to get the data for relatively prolonged cognitive tasks it is necessary to design the task so as to report time benchmarks. In general, to yield interpretable results, researchers must have very good control over what the participant is doing cognitively. Techniques include timing of presentations of information, giving task instructions re the strategy participants should use, or selecting participants according to the strategies they can be shown to be using. The very best of experimental psychology is called for. Sharing this type of experimental know-how would assure the quality of research that is supported and to accelerate progress.

EEG recording is attractive because signals can be recorded as the brain activity is occurring, at the same time scale, in contrast to other imaging technologies. EEG recording has many advantages: speed, relatively low cost, and practicality for use for use even in classroom settings. For studies of the effects of classroom climate, as suggested by Cobb, or for distinguishing between deep and shallow engagement in instruction, as Koedinger suggested, or for large-scale studies of the amount of practice required to achieve automated skills, this is likely to be the method of choice. There have been great technical advances in making many electrode recording caps with standard locations and less time consuming application to the research participant, and telemetering EEG signals rather than tethering subjects to electrode wires. EEG does not however, provide good information on localization of a signal within the brain. Several computer programs have been developed to solve this inverse problem by adding the information about where one expects to find brain activity or changes in brain activity. That information might be provided, for example, by prior fMRI research. One can anticipate continuing advances in the ability to merge data from various imaging technologies with that of EEG results. In parallel, further efforts to improve localization of EEG source areas are needed.

Computational methods. Computational cognitive modeling, supported by a rich research background, can support the alignment of observed brain activity with cognitive activity. Rigorous cognitive task analysis of what the participant must be doing to accomplish the experimental task is necessary. This analysis can be further refined and tested by computational modeling of cognition. Computational modeling of cognition also brings us closer to the modern thinking about how the brain does cognition or, indeed, functions in general. The brain is composed of many dynamically interacting neural networks. Many different areas of the brain are somehow working in concert to perform its various functions. Several workshop participants, notably Vinod Menon and Peter Bergethon, touched upon this way of thinking about brain function. Understanding how these computational mechanisms of the brain really work, how they might be linked to computational modeling at the higher level of description, is still very much a frontier issue. Training opportunities have for optimal uses of cognitive modeling approaches and sharing this type of experimental know-how would also enhance the quality of research and accelerate progress.

Terminology. The terminology identifying different parts of the brain presents some significant barriers. It seems that each bit of the human brain has at least four different labels that are in common use in neuroscience. This is a lot to learn for researchers from other fields. It is a significant cognitive barrier to reviewing and combining the results from researchers who prefer to use different systems of terminology. Perhaps a computerized visualization tool could aid researchers in doing this thinking and communication. The situation becomes even more complex with cross-species research where it is uncertain which brain areas are really analogous and how comparable the functions of apparently analogous areas actually are.

Research teams: The excitement of modern techniques for imaging brain activity is that they enable us to study the large and complex brain of the human who can do learn and do mathematics, learn and use language, in a detailed and systematic way for the first time. But we still have much to learn about how to do that effectively. Multi-disciplinary teams are needed to do this research effectively.

Appendix A

References and Readings

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Appendix B

Workshop on Neuroscience and Mathematical Cognition

Identifying Gaps to Bridge

May 17-18, 2007

National Science Foundation

Stafford I, Room 375

Thursday, May 17, 5:30pm – 7:30pm

Westin Arlington Gateway Hotel - Pinzimini

801 North Glebe Road

Arlington, Virginia 22203

5:30 – 6:00 Orientation (Logistics and Introduction to the Charge)

6:00 – 7:30 Reception Hosted by Dr John Bruer, McDonnell Fioundation

Friday, May 18, 7:30am - ~6:00pm

National Science Foundation, Stafford I, Room 375

4201 Wilson Boulevard

Arlington, Virginia 22230

Welcome and Workshop Overview

7:30 – 8:00 Arrive at NSF Lobby

8:00 – 8:15 Welcome: Dr. Arden Bement, Director, NSF

8:15 – 8:30 Workshop Overview: Dr. John Anderson, CMU

8:30 – 8:45 Presentation: Dr. Liz Brannon, Duke

Presentation of Homework Assignments

8:44 – 9:00	Dr. Peter Bergethon, Boston University
9:00 – 9:15	Dr. John Bruer, McDonnell Foundation
9:15 – 9:30	Dr. Paul Cobb, Vanderbilt
9:30 – 9:45	Dr. Rochel Gelman, Rutgers
9:45 – 10:00	Dr. Layne Kalbfleisch, George Mason University

Coffee Break and Light Refreshments

10:00 – 10:15	Light Refreshments

Presentations of Homework Assignments (continued)

10:15 – 10:30	Dr. Ken Koedinger, CMU
10:30 – 10:45	William McCallum, University of Arizona
10:45 – 11:00	Dr. Vinod Menon, Stanford University
11:00 – 11:15	Dr. Sharlene Newman, Indiana University
11:15 – 11:30	Dr. Russell Poldrack, UCLA
11:30 – 11:45	Dr. Curtis Tatsuoka, Cleveland Clinic and Cleveland Clinic Lerner School of Medicine of Case Western Reserve University
11:45 – 12:00	Dr.Alan Yuille, UCLA

Working Lunch and Break-out Groups

Noon – 1:30	Working Lunch and Break-out Groups

Break-outs Groups (continued)

2:00 - ~4:00	Break-out Groups

Report-out to the National Science Foundation

~4:00 - ~5:30	Report to NSF with Dr. Arden Bement, Director; Kathie Olsen, Deputy Director, Associate Directors and NSF staff.

Appendix C
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Appendix D

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Appendix E

Acknowledgements and Figures

The scientists and students of mathematics thank Dr. John Bruer, President of the McDonnell Foundation for continuing study, support and interest in the generation of new knowledge through its support of research and scholarship, especially in the area of mathematical teaching, learning and policy. Support in aid of the present meeting is much appreciated.

Title Page

This image shows a subset (13% of 254,474) fiber pathways in a human brain using MRI based diffusion weighted imaging (DWI) techniques to show anatomical connectivity in the brain. The technique quantifies the amount of water diffusion in each 2mm cubic voxel of the human brain. The diffusion pattern at each voxel is classified as being in fibers passing in 1 to 3 directions indicative of a fiber pathway or crossing fiber pathways or omnidirectional (uniformily) as in grey matter. This provides directional tensors that produce stream lines allowing reconstruction of the fiber pathway from source to destination throughout the brain. The rendering shows the fibers passing through two 2mm saggital planes +/16 mm from the midline. The colors indicate the direction of the fiber green forward/back, blue bottom/top, and red left/right. The data were collected by Walter Schneider Department of Psychology and Learning Research and Development Center at the University of Pittsburgh. The data were collected using a 256-direction DWI scan on a 3.0 T Siemens Trio, with a 32-channel receiver coil. Fiber Tractography of Diffusion Weighted images were performed using the Diffusion Toolkit and visualized in TrackVis (Ruopeng Wang, Van J. Wedeen, TrackVis.org).

Figure 1, p. 4

Figure 1 depicts the left and right side of the human brain. Marked in color are three brain areas in the parietal cortex, thought to be involved in performing various aspects of mathematical tasks (See text for details). Stanislas Dehaene, Mauela Piazza, Philippe Pinel, and Laurent Cohen, "Three Parietal Circuits for Number Processing," Cognitive Neuropsychology, 2003, 20 (3/4/5/6) pp. 487-506, Taylor & Francis. (re-printed by permission of the publisher (Taylor & Francis Ltd, http://www.informaworld.com).

Figure 2, p. 5

John Anderson, Carnegie Mellon University.

Figure 2 shows the mean percent change in BOLD signal in the left lateral inferior prefrontal cortex in the predicted and observed effects of practice solving linear equation problems on brain activity in the prefrontal cortex . (Separate curves are shown for students performing simple and complex algebraic transformations early (red) and late (blue) in the training protocol. (See text for further details).

Figure 3A and 3B, p. 11

Elizabeth Brannon, Department of Psychology and Neuroscience, Duke University.

Figure 3 shows the upper panel (A) shows the correlation between performance on number tasks and fractional anisotropy (FA) values in children and adults. (B) The accompanying brain image of the brain shows tracts in children colorized as follows: in orange are callosal fiber tracts connecting parietal cortices (Isthmus); in green are callosal fiber tracts connecting frontal cortices (Genu); in blue are callosal fiber tracts connecting occipital cortices (posterior splenium Cantlon, Brannon, and Pelphrey, unpublished work in the laboratory; manuscript in preparation. (See text for further details).